Test 1 / Numerical Mathematics 1 / May 9th 2022, University of Groningen

Instructions

- You have 2 hours to complete the test. When applicable, people with special facilities have 2h20 minutes in total.
- The exam is "closed book", meaning that you can only make use of the material given to you.
- The grade will be computed as the number of obtained points, plus 1.
- All answers need to be justified using mathematical arguments.

Questions

Consider an arbitrary scalar function $g \in C^5([a,b])$. We define a numerical integration method over an interval $[a,b]$ by:

$$\tilde{I}(g) = \frac{b-a}{2}[g(a)+g(b)] + \frac{(b-a)^2}{12}[g'(a)-g'(b)]$$

(a) $\boxed{1.5}$ Suppose that values of $g(x_k), g'(x_k)$ are available at distinct nodes $x_0, \dots, x_n$. Obtain a formula for a composite numerical integration on the nodes $x_0, \dots, x_n$, with a uniform spacing $x_{j+1} - x_j = h$, based on $\tilde{I}(g)$.

At each subinterval $[x_j, x_{j+1}]$ the quadrature looks like: **(0.75 pt)**

$$\frac{h}{2}[g(x_j)+g(x_{j+1})] + \frac{h^2}{12}[g'(x_j)-g'(x_{j+1})]$$

and adding up over all intervals **(0.75 pt)**

$$\sum_{j=0}^{n-1} \frac{h}{2}[g(x_j)+g(x_{j+1})] + \frac{h^2}{12}[g'(x_j)-g'(x_{j+1})]$$

(b) $\boxed{4.5}$ Prove that $\tilde{I}(f)$ has degree of exactness 3.

First, let us check that ALL cubic polynomials can be integrated exactly **(0.75 pt)**:

$$q(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \rightarrow \int_a^b q(x)dx = \alpha_1(b-a) + \alpha_2(\frac{b^2}{2} - \frac{a^2}{2}) + \alpha_3(\frac{b^3}{3} - \frac{a^3}{3}) + \alpha_4(\frac{b^4}{4} - \frac{a^4}{4})$$

while **(0.75 pt)**

$$\begin{aligned} \tilde{I}(q) &= \frac{b-a}{2}[\alpha_1 + \alpha_2 a + \alpha_3 a^2 + \alpha_4 a^3 + \alpha_1 + \alpha_2 b + \alpha_3 b^2 + \alpha_4 b^3] \\ &+ \frac{(b-a)^2}{12}[\alpha_2 + 2\alpha_3 a + 3\alpha_4 a^2 - \alpha_2 - 2\alpha_3 b - 3\alpha_4 b^2] \end{aligned}$$

and **(0.75 pt)**

$$\begin{aligned} \tilde{I}(q) &= (b-a)\alpha_1 + \alpha_2(\frac{b^2}{2} - \frac{a^2}{2}) + \frac{b-a}{2}\alpha_3[a^2 + b^2] + \frac{b-a}{2}\alpha_4[a^3 + b^3] \\ &+ \frac{(b-a)^2}{6}\alpha_3[a-b] + \frac{(b-a)^2}{4}\alpha_4[a^2 - b^2] \end{aligned}$$

and working out the α_3-term **(0.75 pt)** and the α_4-term **(0.75 pt)**, and comparing with the exact integral, shows that cubic polynomials are integrated exactly. Finally, we have to show that one polynomial of order 4 is not integrated exactly, on a specific interval $[a, b]$. For instance, take x^4 over $[0, 1]$, and the result of the integral is one **(0.25 pt)**. The result of the numerical integration leads to: **(0.5 pt)**

$$\frac{1}{2}[0^4 + 1^4] + \frac{(1)^2}{12}[4 \cdot 0^3 - 4 \cdot 1^3] = 1/2 - 4/12 = 1/6$$

finalizing the proof.

(c) [3] Suppose that values of $g(x_k), g'(x_k)$ are given at distinct nodes $x_0, \ldots, x_n$. Write the system of equations for the coefficients of an approximating polynomial of degree $2n+1$ such that the function values and its derivatives match with the ones of the original function at $x_0, \ldots, x_n$. Obtain explicitly the form of the approximating polynomial for $n = 1$ assuming $x_0 = 0$ and $x_1 = 1$. What is the error between the original and approximating polynomial if the original one is of degree 1?

Define the approximating polynomial as $\alpha_0 + \alpha_1 x + \cdots + \alpha_{2n+1} x^{2n+1}$ **(0.25 pt)**. At each node x_k we can obtain a systems of equations:

$$\begin{aligned} g(x_k) &= \alpha_0 + \alpha_1 x_k + \cdots + \alpha_{2n+1} x_k^{2n+1}, k = 0, \ldots, n \quad \textbf{(0.25 pt)} \\ g'(x_k) &= \alpha_1 + \cdots + (2n+1)\alpha_{2n+1} x_k^{2n}, k = 0, \ldots, n \quad \textbf{(0.25 pt)} \end{aligned}$$

For $n = 1$ with $x_0 = 0$ and $x_1 = 1$, we write explicitly the system of equations:

$$\begin{aligned} g(0) &= \alpha_0 \quad \textbf{(0.25 pt)} \\ g'(0) &= \alpha_1 \quad \textbf{(0.25 pt)} \\ g(1) &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \quad \textbf{(0.25 pt)} \\ g'(1) &= \alpha_1 + 2\alpha_2 + 3\alpha_3 \quad \textbf{(0.25 pt)} \end{aligned}$$

where then we need to solve for the coefficients α_2, α_3:

$$\begin{aligned} \alpha_3 &= g'(1) + g'(0) - 2g(1) + 2g(0) \quad \textbf{(0.25 pt)} \\ \alpha_2 &= -g'(1) - 2g'(0) + 3g(1) - 3g(0) \quad \textbf{(0.25 pt)} \end{aligned}$$

So, in the case that a linear polynomial is approximated, this means $g'(1) = g'(0) = g(1) - g(0)$ **(0.25 pt)**, and hence $\alpha_2 = \alpha_3 = 0$ **(0.25 pt)**. Therefore, the approximating polynomial is $f(0) + f'(0)x$ leading to an exact approximation (zero error) **(0.25 pt)**.